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VII. PROCESSING AND TRANSMISSION OF INFORMATION

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A. CONSTRAINTS IMPOSED BY THE LAW OF CONSERVATION OF ENERGY ON POSSIBLE FORMS OF INTERACTION HAMILTONIANS

National Aeronautics and Space Administration (Grant NGL 22-009-013)

V. Chan

1. Introduction

A popular theory of measurement of quantum systems says that if we describe states of a quantum system S as elements in a Hilbert space $\mathcal{H}_S$, then any measurement can be characterized by a Hermitian operator in Hilbert space $\mathcal{H}_S$. This notion of quantum measurement is too restricted, and we shall consider measurements characterized by Hermitian operators in extended Hilbert spaces that include the original space $\mathcal{H}_S$. One possible way of implementing such measurement is to let an apparatus A interact with the system S and then perform subsequent measurement on the combined system S+A.[1-3] Therefore we want to know what types of interaction are feasible and, in particular, to find restrictions on possible types of interaction when we invoke some form of conservation law. We know that we can characterize interactions between two quantum systems by specifying[1, 2] the interaction Hamiltonian H_I. In this report we discuss the restriction of the law of conservation of energy on the allowable form of H_I.

2. States of S+A Described by Pure States

Let us assume that before a certain contact time t_c the two systems S and A are noninteracting and evolve independently according to their individual free Hamiltonians, $\mathcal{H}_s$ and $\mathcal{H}_A$, respectively. The Hamiltonian for the combined system S+A before t_c is then $H = H_s \otimes I_A + I_s \otimes H_A$, where I_A and I_s are the identity operators in $\mathcal{H}_A$ and $\mathcal{H}_s$, respectively. Let $\mathcal{M}_{S+A}$ be a complete linear subspace of $\mathcal{H}_s \otimes \mathcal{H}_A$ such that for $t > t_c$ the states of S+A described by vectors in $\mathcal{M}_{S+A}$ are noninteracting. Let $|a^o+s^o\rangle\rangle \in \mathcal{M}_{S+A}$ be the Schrödinger state of S+A at t_c. The energy of the combined

system at this point is

$$E^{o}_{S+A} = \langle\langle s^{o}+a^{o}|H|a^{o}+s^{o}\rangle\rangle.$$

For $t > t_c$, $|a^t+s^t\rangle\rangle = U_t|a^o+s^o\rangle\rangle$, where $U_t \equiv \exp\{-\frac{i}{\hbar} H'(t-t_c)\}$.
Then for $t > t_c$,

$$E^{t}_{S+A} = \langle\langle s^{t}+a^{t}|H'|a^{t}+s^{t}\rangle\rangle$$

$$= \langle\langle s^{o}+a^{o}|U_t^{\dagger}H'U_t|a^{o}+s^{o}\rangle\rangle.$$

Since H' is the generator of the unity group U_t, it commutes with U_t; that is, the commutator $[H', U_t] = 0$. Therefore, for $t > t_c$,

$$E^{t}_{S+A} = \langle\langle s^{o}+a^{o}|H'|a^{o}+s^{o}\rangle\rangle$$

$$= \langle\langle s^{o}+a^{o}|H|a^{o}+s^{o}\rangle\rangle + \langle\langle s^{o}+a^{o}|H_I|a^{o}+s^{o}\rangle\rangle$$

$$= E^{o}_{S+A} + \langle\langle s^{o}+a^{o}|H_I|a^{o}+s^{o}\rangle\rangle.$$

The law of conservation of energy requires

$$E^{t}_{S+A} = E^{o}_{S+A}; \qquad \text{for all } t.$$

Hence this implies (*) $\langle\langle s^{o}+a^{o}|H_I|a^{o}+s^{o}\rangle\rangle = 0$.

A sufficient (but not necessary) condition for H_I to satisfy the constraint (*) is to have $|a^o+s^o\rangle\rangle$ in the null space ($\mathcal{N}_{H_I}$) of H_I (condition 1a). Or equivalently if we have already specified $\mathcal{M}_{S+A}$, we require H_I to be representable completely by vectors in $\mathcal{M}^{\perp}_{S+A}$, where $\mathcal{M}^{\perp}_{S+A} \equiv \mathcal{H}_s \otimes \mathcal{H}_A - \mathcal{M}_{S+A}$ is the orthogonal space of $\mathcal{M}_{S+A}$ (condition 1b).

A necessary and sufficient condition can be found. Let $H_I\mathcal{M}_{S+A} \equiv \{\text{all } x \in \mathcal{H}_s \otimes \mathcal{H}_A: x = H_I y, \text{ some } y \in \mathcal{M}_{S+A}\}$. Let $\mathcal{M}^{H_I}_{S+A}$ be the completion of this space. In other words, $\mathcal{M}^{H_I}_{S+A}$ is the range space of H_I with domain restricted to $\mathcal{M}_{S+A}$. Then a necessary and sufficient condition for (*) to hold is $\mathcal{M}^{H_I}_{S+A} \subset \mathcal{M}^{\perp}_{S+A}$ (condition 2).

3. States of S+A Described by Density Operators

Let the state of S+A at t_{c_-} be described by the density operator ρ^o_{S+A}. Requiring no interaction between S and A before t_c means that ρ^o_{S+A} can be represented completely by vectors in $\mathscr{M}_{S+A}$. For $t > t_c$ the density of the combined system is

$$\rho^t_{S+A} = U_t \rho^o_{S+A} U_t^\dagger.$$

The mean energy for $t > t_c$ is

$$E^t_{S+A} = \mathrm{Tr}\left\{\rho^t_{S+A} H'\right\}$$

$$= \mathrm{Tr}\left\{U_t \rho^o_{S+A} U_t^\dagger H'\right\},$$

where Tr denotes the trace.

Since $[H', U_t] = 0$ and $U_t^\dagger$ commute with H',

$$E^t_{S+A} = \mathrm{Tr}\left\{U_t \rho^o_{S+A} H' U_t^\dagger\right\}.$$

Since unitary transformations do not change the trace of operators,

$$E^t_{S+A} = \mathrm{Tr}\left\{\rho^o_{S+A} H'\right\}$$

$$= \mathrm{Tr}\left\{\rho^o_{S+A} H\right\} + \mathrm{Tr}\left\{\rho^o_{S+A} H_I\right\}$$

$$= E^o_{S+A} + \mathrm{Tr}\left\{\rho^o_{S+A} H_I\right\}.$$

Conservation of energy requires that, for all t, $E^t_{S+A} = E^o_{S+A}$, which implies

$$(**) \quad \mathrm{Tr}\left\{\rho^o_{S+A} H_I\right\} = 0.$$

Let $\mathcal{H}_o$ be the Hilbert space of bounded Hermitian operators defined on $\mathcal{H}_S \otimes \mathcal{H}_A$ and the inner product in $\mathcal{H}_o$ be defined as $(A, B) = \mathrm{Tr}\{A, B\}$ for all $A, B \in \mathcal{H}_o$. Then a necessary and sufficient condition to satisfy (**) is

$$\rho^o_{S+A} \in \mathcal{H}^\perp_{H_I} = \mathcal{H}_o - \{H_I\} \qquad \text{(condition 3a)},$$

where $\{H_I\}$ is the subspace generated by H_I, and $\mathcal{H}^\perp_{H_I}$ is the subspace orthogonal to it.

Furthermore, if we have already specified the possible choices of ρ^{o}_{S+A} by requiring that they be representable by vectors in $\mathscr{M}_{S+A}$ and denoted the subspace of $\mathcal{H}_o$ generated by these sets of possible density operators by $\mathcal{H}_{\rho_{S+A};\mathscr{M}_{S+A}}$, then an equivalent condition of (3a) is

$$H_I \in \mathcal{H}^{\perp}_{\rho_{S+A};\mathscr{M}_{S+A}} \equiv \mathcal{H}_o - \mathcal{H}_{\rho_{S+A};\mathscr{M}_{S+A}} \qquad \text{(condition 3b).}$$

A more illuminating sufficient (but not necessary) condition is easily found by noting that if we require H_I to be representable by vectors in $\mathscr{M}^{\perp}_{S+A}$ the condition (**) is always satisfied (condition 4).

References

1. V. Chan, "Interaction Formulation of Quantum Communication Theory," Quarterly Progress Report No. 106, Research Laboratory of Electronics, M. I. T., July 15, 1972, pp. 128-132.
2. V. Chan, "Sequential Detection of Signals Transmitted by a Quantum System (Equiprobable Binary Pure State)," Quarterly Progress Report No. 107, Research Laboratory of Electronics, M. I. T., October 15, 1972, pp. 89-93.
3. V. Chan, "Statistically Dependent Quantum Systems: The Paradox of Einstein, Podolsky, and Rosen," Quarterly Progress Report No. 108, Research Laboratory of Electronics, M. I. T., January 15, 1973, pp. 225-229.

B. REALIZATION OF AN OPTIMUM QUANTUM MEASUREMENT BY EXTENSION OF HILBERT-SPACE TECHNIQUE

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V. Chan

A system S is in one of M equiprobable pure states $\{|f_i\rangle\}_{i=1}^{M}$ and these states are linearly dependent with certain symmetry such that $\sum_{i=1}^{M} b|f_i\rangle\langle f_i| = I_s$, $b > 0$, where I_s is the identity operator in Hilbert space $\mathcal{H}_s$ that describes the system S. To maximize the probability of correct detection, we want to observe S and determine in which of M states it is. It is known that there is a solution to this problem.[1]

The measurement operators are Hermitian and positive definite: $Q_i = b|f_i\rangle\langle f_i|$, $i = 1, \ldots, M$. These Q_i are not orthogonal in general, however, and do not correspond to any measurement on S alone that can be described by a Hermitian operator in $\mathcal{H}_s$. Helstrom and Kennedy[2] have proposed to synthesize these types of measurement by bringing into consideration another quantum system A, called the apparatus, described

by Hilbert space $\mathcal{H}_A$ so that there exists a set of orthogonal positive definite Hermitian measurement operators $\{\pi_i\}_{i=1}^{M}$ in the tensor product Hilbert space $\mathcal{H}_s \otimes \mathcal{H}_A$ and a density operator ρ_A for A with $Q_i = \mathrm{Tr}_A\{\pi_i \rho_A\}$, where Tr_A indicates taking a partial trace over $\mathcal{H}_A$. Now the π_i correspond to a Hermitian measurement on $\mathcal{H}_s \otimes \mathcal{H}_A$. We have solved this problem for a particular case with M = 3.

If the possible states of S are the three shown in Fig. VII-1, it can be shown[1] that $\sum_{i=1}^{3} \frac{2}{3} |s_i\rangle\langle s_i| = I_s$, so that $Q_i = \frac{2}{3} |s_i\rangle\langle s_i|$, $i = 1, \dots, 3$, and the probability of correct detection is $P[C] = \frac{1}{3} \sum_i \mathrm{Tr}\,[\rho_i Q_i] = \frac{2}{3}$.

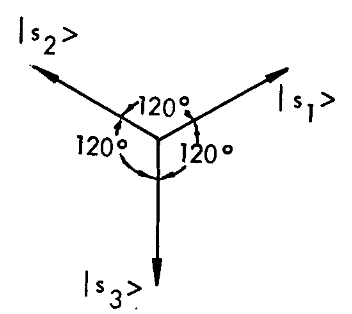

Fig. VII-1. Possible states of S.

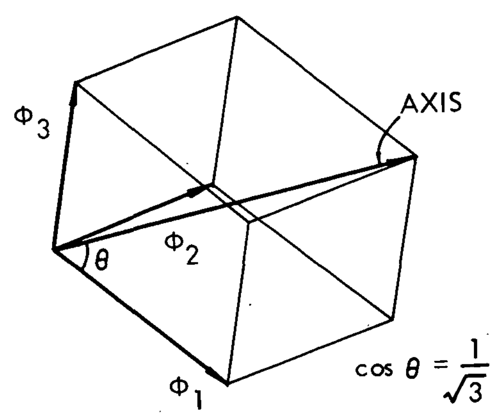

Fig. VII-2. Configurations of $\pi_i' = |\phi_i\rangle\langle\phi_i|$.

Pick any apparatus A described by Hilbert space $\mathcal{H}_A$ of dimension $N \geq 2$ (hence the dimension of $\mathcal{H}_s \otimes \mathcal{H}_A \geq 4$). Let $\rho_A = |a\rangle\langle a|$, where $|a\rangle$ is any pure state. Therefore the three possible joint states of S+A are $\{|s_i\rangle|a\rangle\}_{i=1}^{3}$, and again they span a two-dimensional subspace in $\mathcal{H}_s \otimes \mathcal{H}_A$, namely $\mathcal{H}_s \otimes \mathcal{M}_{|a\rangle}$, where $\mathcal{M}_{|a\rangle}$ is the subspace generated by $|a\rangle$. Choose any other one-dimensional subspace $\mathcal{M}_{S+A}$ in $\mathcal{H}_s \otimes \mathcal{M}^{\perp}_{|a\rangle}$, where $\mathcal{M}^{\perp}_{|a\rangle}$ is the orthogonal subspace of $\mathcal{M}_{|a\rangle}$. Then three orthogonal measurement operators $\{\pi_i'\}_{i=1}^{3}$ can be found in $\mathcal{H}_s \otimes \mathcal{M}_{|a\rangle} \cup \mathcal{M}_{S+A}$ to satisfy our requirements.

We shall show that $\pi_i' = |\phi_i\rangle\langle\phi_i|$, where the $|\phi_i\rangle$ are orthonormal vectors. By symmetry considerations, it is clear that we want the axis of the coordinate system made up of $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$ to be perpendicular to the plane spanned by the $|s_i\rangle$, and the projections of the $|\phi_i\rangle$ on the plane of the $|s_i\rangle$ along the axis should coincide with these respective $|s_i\rangle$, so that $|\langle\phi_i|s_i\rangle|$ = a constant for all i is maximized (see

Fig. VII-2). By straightforward geometric calculations $|\langle \phi_i | s_i \rangle|^2 = \frac{2}{3}$ so that $P[C] = \frac{1}{3} \sum_{i=1}^{3} Tr\{\rho_i \pi_i'\} = \frac{2}{3}$. *Hence, the* $|\phi_i\rangle$ *are indeed optimum and if we wish to require* in addition that the sum of the measurement operators equals the identity operator in $\mathcal{H}_S \otimes \mathcal{H}_A$, we need only define $\pi_i = \pi_i' \otimes I_d$, where I_d is the identity operator of $\mathcal{H}_S \otimes \mathcal{H}_A - \{\mathcal{H}_S \otimes \mathcal{M}\,|a\rangle \cup \mathcal{M}_{S+A}\}$, then $\sum_{i=1}^{3} \pi_i = I_{S+A}$.

References

1. A. S. Holevo, "Statistical Problems in Quantum Physics," unpublished manuscript, Steklov Mathematical Institute, Moscow, U.S.S.R., to appear in Proc. Soviet-Japanese Symposium on Probability and Statistics, 1972.
2. C. W. Helstrom and R. S. Kennedy, "Noncommuting Observables in Quantum Detection and Estimation Theory" (submitted to IEEE Trans. on Information Theory).

C. ON THE OPTIMUM QUANTUM RECEIVER FOR THE M-ARY LINEARLY INDEPENDENT PURE STATE PROBLEM

National Aeronautics and Space Administration (Grant NGL 22-009-013)

R. S. Kennedy

It has been conjectured that the optimum quantum receiver for digital communication may not always be characterized by a set of (commuting) observables on the system space, i.e., a set of commuting Hermitian operators with complete sets of eigenvectors.[1-6] Recently, it has been demonstrated that this conjecture is, in fact, true[7-9] and there has been renewed interest in delineating conditions for which the optimum receiver can be characterized by observables.

It is known that the optimum receiver for binary signaling is characterized by an observable.[10, 3, 5] It has also been shown that, among those receivers characterized by an overcomplete set of measurement states, the optimum receiver for the M-ary linearly independent pure-state problem can be characterized by an observable.[11, 2] Not all quantum measurements can be characterized by such overcomplete sets,[8,9] however, and the characterization of the optimum receiver for the M-ary problem has remained open. In this report we show that the optimum quantum receiver for the M-ary pure state problem can, in fact, be characterized by an observable when the M pure states are linearly independent.

The problem of interest can be stated as follows. One of a set of M messages is transmitted, the i^{th} message occurring with (nonzero) a priori probability p_i. The transmission of message i causes the field at the receiver to be in the quantum state $|u_i\rangle$. That is, there is no randomness in the channel nor is there any additive noise.

It has been shown that any quantum receiver for M-ary digital communication can be characterized by a set of M nonnegative definite Hermitian operators, π_i, that sum to the identity operator.[8,9] That is,

$$\pi_i \geq 0 \qquad i = 1, \ldots, M \tag{1a}$$

$$\sum_{i=1}^{M} \pi_i = I. \tag{1b}$$

It has further been shown that the necessary and sufficient conditions for a set of π_i satisfying (1) to minimize the error probability are, in addition to (1),[3,5,8]

$$\sum_i p_i(\rho_i\pi_i - \pi_i\rho_i) = 0 \tag{2a}$$

$$\Pi_j\Big(\sum_i p_i\pi_i\rho_i - p_j\rho_j\Big) = 0 \tag{2b}$$

$$\sum_i p_i\pi_i\rho_i - p_j\rho_j \geq 0, \quad \text{all } j. \tag{2c}$$

Here ρ_i is the density operator of the received field when message i is transmitted.

For the pure-state problem, which is of interest here, the density operators ρ_i are given by

$$\rho_i = |u_i\rangle\langle u_i| \qquad i = 1, \ldots, M. \tag{3}$$

Since we have assumed that the $|u_i\rangle$ are linearly independent, their weighted sum will vanish only if all of the weighting coefficients vanish. That is,

$$\sum a_i|u_i\rangle = 0 \tag{4}$$

implies that all of the a_i are zero. Although (3) and (4) yield some simplification of (2), they do not permit an explicit solution for the π_i.[11,3,5] As we shall show, they do imply that π_i satisfying (2) can be found which describe a receiver that is characterized by observables. Precisely stated, if the Hilbert space of the system is taken to be that spanned by the $|u_i\rangle$, the operators π_i that satisfy (2) subject to (1) are computing projection operators. That is,

$$\pi_i\pi_j = \delta_{ij}\pi_i \qquad i, j = 1, \ldots, M, \tag{5}$$

where the δ_{ij} is the Kronecker delta; $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$.

Of course, there is no a priori reason for limiting the space of the system to be that

spanned by the $|u_i\rangle$. Moreover, if larger spaces are considered, (2) will have solutions that do not satisfy (5). It can be shown, however, that the use of a larger space does not lead to performance improvement. That is, insofar as the performance is concerned, no generality is lost by assuming that the system space is spanned by the $|u_i\rangle$. The use of a larger space may, however, lead to simpler, and more easily interpreted, measurements. We assume that the Hilbert space associated with the system is spanned by the M linearly independent vectors $|u_i\rangle$ and prove that the π_i also satisfy (5).

The proof has three parts. First, we demonstrate that a set of vectors $|f_i\rangle$, i = 1, . . . , M can be found for which

$$\langle f_i u_j\rangle = 0 \quad i \neq j, \quad i, j = 1, \ldots, M \tag{6a}$$

$$\langle f_i u_i\rangle \neq 0 \quad i = 1, \ldots, M. \tag{6b}$$

Using these vectors, we then show that the vectors $|v_i\rangle$ defined by

$$|v_i\rangle \equiv \pi_i |u_i\rangle \tag{7}$$

satisfy the expression

$$\pi_j |v_i\rangle = \delta_{ij} |v_i\rangle \qquad i, j = 1, \ldots, M, \tag{8}$$

and are linearly independent when the π_j satisfy (2). The validity of (5) follows easily from (8) and the linear independence of the $|v_i\rangle$.

To demonstrate the existence of M vectors $|f_i\rangle$ satisfying (6), we invoke the assumptions that the $|u_i\rangle$ are linearly independent and span the space of the system. These assumptions imply that, for every i, there exists a vector $|f_i\rangle$ that is orthogonal to $u_j\rangle$ for all $j \neq i$ and is not orthogonal to $|u_i\rangle$. That is, $|f_i\rangle$ is orthogonal to the M−1 dimensional space spanned by $|u_j\rangle$, $j \neq i$. Thus (6) is proved.

To prove (8) we observe that, for any set π_i satisfying (2b),

$$\pi_j(\sum_i p_i\pi_i\rho_i - p_j\rho_j) |f_k\rangle = 0 \qquad j, k = 1, \ldots, M. \tag{9a}$$

But

$$\rho_i |f_k\rangle = u_i\rangle\langle u_i f_k\rangle = \delta_{ik} C_i |u_i\rangle, \tag{9b}$$

with $C_k \neq 0$, the rightmost equality being a consequence of (6). Use of (9b) to eliminate the density operators from (9a) yields

$$\pi_j\left(\sum_i p_i\pi_i\delta_{ik}C_i\,|u_i\rangle\right) - p_j\pi_j\delta_{jk}C_j|u_j\rangle = 0 \qquad j, k = 1, \ldots, M, \tag{10a}$$

or

$$p_kC_k\pi_j\pi_k|u_k\rangle = \delta_{jk}p_jC_j\pi_j|u_j\rangle \qquad j, k = 1, \ldots, M. \tag{10b}$$

Introducing the vectors $|v_i\rangle$, defined by (7), in (10b) and noting from (6b) and (9b) that the C_j are nonzero, we obtain (8).

To prove that the $|v_i\rangle$ are linearly independent, we suppose, to the contrary, that they are linearly dependent. Then the space contains a vector, say $|g\rangle$, other than the null vector such that

$$\langle v_ig\rangle = 0 \qquad i = 1, \ldots, M. \tag{11}$$

For this vector it follows from (2c) that

$$\sum_i \langle gp_i\pi_i\rho_ig\rangle - \langle gp_j\rho_jg\rangle \geq 0 \qquad j = 1, \ldots, M. \tag{12}$$

But

$$\pi_i\rho_i|g\rangle = \pi_i|u_i\rangle\langle u_ig\rangle = |v_i\rangle\langle u_ig\rangle \qquad i = 1, \ldots, M,$$

the rightmost equality being a consequence of (7). Thus, by virtue of the assumed condition (11),

$$\langle g\pi_i\rho_ig\rangle = \langle gv_i\rangle\langle u_i g\rangle = 0 \qquad i = 1, \ldots, M. \tag{13}$$

Therefore, for (12) and hence (2c) to be satisfied, it is necessary that $\langle g\rho_jg\rangle$ vanish for all j. That is,

$$\langle gu_j\rangle = 0 \qquad j = 1, \ldots, M. \tag{14}$$

But since the $|u_j\rangle$ span the space, (14) implies that $|g\rangle$ is the null vector. Consequently the $|v_i\rangle$ cannot be linearly dependent.

The proof is completed by noting that, since the M vectors $|v_i\rangle$ are linearly independent, any vector $|w\rangle$ in the system space may be expressed as

$$|w\rangle = \sum_i b_i|v_i\rangle. \tag{15}$$

Thus

$$\pi_j|w\rangle = \sum_i b_i\pi_j|v_i\rangle = \sum_i b_i\delta_{ij}|v_i\rangle = b_j|v_j\rangle, \tag{16}$$

the middle equality being a consequence of (8). Also

$$\pi_k\pi_j|w\rangle = \pi_k\{b_j|v_j\rangle\} = b_j\pi_k|v_j\rangle = b_j\delta_{jk}|v_j\rangle = \delta_{jk}\pi_j|w\rangle, \tag{17}$$

where the middle equality follows from (8) and the rightmost equality follows from (16). Since (17) must be true for all vectors $|w\rangle$, we conclude that

$$\pi_k\pi_j = \delta_j k\pi, \quad \text{all } j, k. \tag{18}$$

Summarizing, for the M-ary linearly independent pure state problem phrased on the space spanned by the M states, the π_i associated with the optimum receiver are commuting projection operators.

The author wishes to thank Professor C. W. Helstrom, Dr. H. P. H. Yuen, and Mr. V. Chan for their comments.

References

1. R. S. Kennedy and S. Karp (Eds.), "Optical Space Communication," NASA SP-217, Office of Technology Utilization National Aeronautics and Space Administration, Washington, D.C., 1969, Sec. 3.4, pp. 18-20.

2. J. P. Gordon and W. H. Louisell, "Simultaneous Measurement of Noncommuting Observables," in P. L. Kelley, B. Lax, and P. E. Tannenwald (Eds.), Physics of Quantum Electronics (McGraw-Hill Book Company, New York, 1966), pp. 833-840.

3. H. P. H. Yuen, "Communication Theory of Quantum Systems," Ph.D. Thesis, Department of Electrical Engineering, M.I.T., June 1970; also Technical Report 482, Research Laboratory of Electronics, M.I.T., August 30, 1971, p. 124.

4. S. D. Personick, "Efficient Analog Communication over Quantum Channels," Ph.D. Thesis, Department of Electrical Engineering, M.I.T., December 1969; also Technical Report 477, Research Laboratory of Electronics, M.I.T., May 15, 1970.

5. H. P. H. Yuen, R. S. Kennedy, and M. Lax, "On Optimal Quantum Receivers for Digital Signal Detection," Proc. IEEE 58, 1770-1773 (1970).

6. C. W. Helstrom, Jane W. S. Liu, and J. P. Gordon, "Quantum-Mechanical Communication Theory," Proc. IEEE 58, 1578-1598 (1970).

7. H. P. H. Yuen and M. Lax, "Multiple Parameter Quantum Estimation and Measurement of Non Self-adjoint Operators," a paper presented at IEEE International Symposium on Information Theory, Asilomar, California, January 31, 1972 (to appear in IEEE Trans. on Information Theory).

8. A. S. Holevo, "Statistical Problems in Quantum Physics," unpublished manuscript, Steklov Mathematical Institute, Moscow, U.S.S.R. (to appear in Proc. Soviet-Japanese Symposium on Probability and Statistics, 1972).

9. C. W. Helstrom and R. S. Kennedy, "Noncommuting Observables in Quantum Detection and Estimation Theory" (submitted for publication to IEEE Trans. (IT)).

10. C. W. Helstrom, "Detection Theory and Quantum Mechanics," Inform. Contr. 10, 254-291 (1967).

11. C. W. Helstrom, "Some Quantum-Mechanical Aspects of Optical Communications," in R. S. Kennedy and S. Karp (Eds.), "Optical Space Communication," NASA SP-217, op. cit., pp. 111-120.